Chapter 06 Feedback

06 Feedback Control System Characteristics

The role of error signals to characterize feedback control system performance.

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1 Content and Learning Objectives

The learning objectives of this lesson and exercises.

1. Be aware of the central role of error signals in analysis of control systems.
2. Recognize the improvements afforded by feedback control in reducing system sensitivity to parameter changes, disturbance rejection, and measurement noise attenuation.
3. Understand the differences between controlling the transient response and the steady-state response of a system.
4. Have a sense of the benefits and costs of feedback in the control design process.

Contents

1 Content and Learning Objectives 1
2 The control system 1
3 Error Signal Analysis 5
4 Sensitivity of Control Systems to Parameter Variations 6
5 Disturbance Signals in a Feedback Control System 8

2 The control system

Definition of a Control System

![Control System Diagram]

Figure 1: A control system is defined as an interconnection of components forming a system that will provide a desired system response.
The Target of a Controller for a Dynamic System.

The goal of the control system

- ensure that the control error $e(t)$ is as small as possible ($y(t)$ follow as closely as possible $r(t)$.
- in the presence of not measurable disturbances $d(t)$. In this case the disturbance is not measurable.
- in the presence of uncertainties about the model
- with a control actions $u(t)$ with saturation. In such a case, the system has not a linear behavior any longer, thus this can be considered as a unknown disturbance.

Mathematical model of the feedforward control system

Conceptual decomposition of the system.

- In order to develop formal analysis, the system is described using blocks that represent behavior of a part.
- As important example, the actuator is often integrated in the plant (e.g. the electrical motor that drives a robot),
- therefore the mathematical model of the plant, $G(s)$, usually comprises also the actuator dynamics.

![Figure 2: Mathematical model of the feedforward control.](image)

Feedforward control synthesis

- Let’s consider the error dynamics:

$$E(s) = R(S) - Y(S)$$

- and the equivalent system, that is the system that comprises the controller and the system:

$$G_{eq}(s) = C(s) \cdot G(s)$$

- we define a desired dynamics of the equivalent system, i.e. we design the $G_{eq}(s)$ such as its poles (⇒ system dynamics) and steady state error are within the specifications required.
- if we set the controller model as:

$$C(s) = \frac{G_{eq}(s)}{G(s)}$$

- we obtain:

$$Y(s) = C(s)G(s)R(s) = \frac{G_{ref}(s)}{G(s)} G(s)R(s) = \frac{G_{eq}(s)}{G(s)} G(s)R(s)$$

Property of controller $C(s)$

- The system has the desired poles (i.e. desired dynamics).
- Example: design a feedforward controller for the system:

$$G(s) = \frac{0.01}{s + 0.1}$$

- aiming to reach a settling time of $t_s = 0.1$ sec. and a zero steady state error (i.e. $\lim_{t \to \infty} r(t) = \lim_{t \to \infty} y(t)$).
- Note: the settling time of the system can be computed rearranging the system as:

$$G(s) = \frac{0.01}{s + 0.1} = \frac{0.01}{10 \cdot s + 1} = \frac{K_p}{\tau s + 1}$$

- with $K_p = 0.1$ and $\tau = 10$, therefore $t_s \approx 3\tau = 30$ sec.
Design of controller $C(s)$

The desired $G_{eq}$ should be so that:

- for the dynamic specification:
  \[ \tau = \frac{t_s}{3} = 0.1/3 = 0.03 \]

- and for the steady state specification:
  \[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG_{eq}(s)R(s) \]

- and then, if
  \[ \lim_{s \to 0} G_{eq}(s) = 1 \]

- we have:
  \[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} sR(s) = \lim_{t \to \infty} r(t) \]

- thus:
  \[ G_{eq}(s) = \frac{1}{0.03s + 1} \]

- thus the controller is
  \[ C(s) = \frac{G_{eq}}{G(s)} = \frac{\frac{1}{0.03s + 1}}{\frac{0.01}{s + 0.1}} \]

- and the overall system is:
  \[ Y(s) = C(s)G(s)R(s) = \frac{\frac{1}{0.03s + 1}}{\frac{0.01}{s + 0.1}}R(s) = \frac{0.01}{s + 0.1}R(s) = \frac{1}{0.03s + 1}R(s) \]

Practical limitation to feedforward control design

- The controller operates without having information about the real performance of the output $y(t)$.
- this lack of information makes not zeros the error $e(t)$, because:
  1. the effect of external disturbances $d(t)$
  2. modeling errors on $G(s)$
- these effects makes practically unfeasible the feedforward control.

Effects on disturbances on feedforward control

- The disturbance $D(s)$ sums its action over the input $R(s)$:
  \[ Y(s) = C(s)G(s)R(s) + G(s)D(s) = \frac{G_{eq}(s)}{G(s)}G(s)R(s) + G(s)D(s) = \frac{G_{eq}(s)}{G(s)}G(s)R(s) + G(s)D(s) = \frac{G_{eq}(s)R(s) + G(s)D(s)}{\text{Disturbance effect}} \]
Effects on modeling errors on $G(s)$ on feedforward control

- Let's consider a modeling error on $G(s)$, that is the actual model $G_{act}(s)$ is different from the nominal model $G(s)$, so that:

$$G_{act}(s) = G(s) \cdot \Delta G(s)$$

- The feedforward controller is still defined on nominal model, so that:

$$Y(s) = C(s) G_{act}(s) R(s) = G_{ref}(s) G_{act}(s) R(s) = G_{eq}(s) \Delta G(s) R(s) = G_{eq}(s) \Delta G(s) R(s)$$

Feedforward control design - Assignment 6.1

- Let's consider a system: $G(s) = \frac{5}{s^2 + 10s + 20}$
- With respect a step input with amplitude equal to one, calculate
  1. the percentual overshoot $S_p$,
  2. the settling time $t_s$,
  3. the steady state error.
- Calculate then a feed-forward control that satisfies the specifications $t_s = 0.5$ sec, $S_p = 10\%$ and zero steady state error.
- Simulate the system with Simulink, in the following conditions:
  1. Ideal conditions.
  2. A disturbance step input of $d(t) = 0.1$.
  3. A modeling error such as the real system to be included in the simulation is: $G_{act}(s) = \frac{5}{(s^2 + 10s + 20)(s + 1.1)}$
  4. A parametric error on the model such as the real system to be included in the simulation is:

$$G_{act}(s) = \frac{5}{(s^2 + 10s + 40)}$$

- Write a short reports on the results (Matlab plot with your own comments)

Feedback Control

**Open Loop (Feedforward Control).**

An open-loop (direct) system operates without feedback and directly generates the output in response to an input signal.

**Closed Loop (Feedback Control).**

A closed-loop syste uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is used by the controller to adjust the actuator.

![Closed-loop control system](image)

The feedback control system is shown Figure 4. Despite the cost and increased system complexity, closed-loop feedback control has the following advantages:

- Decreased sensitivity of the system to variations in the parameters of the process.
- Improved rejection of the disturbances.
- Improved measurement noise attenuation.
- Improved reduction of the steady-state error of the system.
- Easy control and adjustment of the transient response of the system.
3 Error Signal Analysis

Error Signal Analysis

- The closed-loop feedback control system has three inputs: \( R(s) \), \( D(s) \) and \( N(s) \), and one output: \( Y(s) \).
- The signals \( D(s) \) and \( N(s) \) are the disturbance and measurement noise signals, respectively.
- Let’s define the tracking error as
  \[
  E(s) = R(s) - Y(s)
  \]
- and we consider an unity feedback: \( H(s) = 1 \).
- It’s straightforward to obtain:
  \[
  Y(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} R(s) + \frac{G(s)}{1 + G(s)C(s)} D(s) - \frac{G(s)C(s)}{1 + G(s)C(s)} N(s)
  \]
- Then recalling the definition of Tracking Error: \( E(s) = R(s) - Y(s) \)
- we have:
  \[
  E(s) = \frac{1}{1 + G(s)C(s)} R(s) - \frac{G(s)}{1 + G(s)C(s)} D(s) + \frac{G(s)C(s)}{1 + G(s)C(s)} N(s)
  \]

Loop Gain

- Let’s define the Loop Gain function:
  \[
  L(s) = G(s)C(s)
  \]
- the Tracking Error can be written as:
  \[
  E(s) = \frac{1}{1 + L(s)} R(s) - \frac{G(s)}{1 + L(s)} D(s) + \frac{L(s)}{1 + L(s)} N(s)
  \]
- Let’s define the Sensitivity function:
  \[
  S(s) = \frac{1}{1 + L(s)}
  \]
- and the Complementary Sensitivity Function:
  \[
  Co(s) = \frac{L(s)}{1 + L(s)}
  \]
- and then:
  \[
  E(s) = S(s) R(s) - S(s) G(s) D(s) + Co(s) N(s)
  \]

- To minimize the tracking error \( E(s) = S(s) R(s) - S(s) G(s) D(s) + Co(s) N(s) \), we want both \( S(s) \) and \( Co(s) \) to be small.
- However \( S(s) \) and \( Co(s) \) are both functions of the controller \( C(s) \), which the control design engineer must select.
- In fact the two are not independent, as it holds:
  \[
  S(s) + Co(s) = 1
  \]
- We cannot simultaneously make \( S(s) \) and \( Co(s) \) small. Obviously, design compromises must be made.
- To analyze the tracking error equation, we need to understand what it means for a transfer function to be “large” or to be “small”.
- The discussion of magnitude of a transfer function is the subject of lesson “Frequency Response Methods”
- Next, we discuss how we can use feedback to reduce the sensitivity of the system to variations and uncertainty in parameters in the process: \( G(s) \).
- This is accomplished by analyzing the tracking error when \( D(s) = N(s) = 0 \).
4 Sensitivity of Control Systems to Parameter Variations

Feedback decreases sensitivity to parameter variation

- A process, represented by the transfer function \( G(s) \), whatever its nature, is subject to a changing environment (aging, etc.)
- In the open-loop system, all these errors and changes result in a changing and inaccurate output.
- A closed-loop system senses the change in the output due to the process changes and attempts to correct the output.
- The sensitivity of a control system to parameter variations is of prime importance.
- A primary advantage of a closed-loop feedback control system is its ability to reduce the system’s sensitivity.

Sensitivity of control system

- Let’s consider the case in which \( D(s) = N(s) = 0 \):
  \[
  Y(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}R(s) = \frac{L(s)}{1 + L(s)}R(s)
  \]
  
  - if \( L(s) \gg 1 \):
    \[
    Y(s) \approx R(s)
    \]
  
  - The output is approximately equal to the input.
  - However, the condition \( L(s) \gg 1 \) may cause the system response to be highly oscillatory and even unstable. But the fact that increasing the magnitude of the loop gain reduces the effect of \( G(s) \) on the output is an exceedingly useful result. Therefore, the first advantage of a feedback system is that the effect of the variation of the parameters of the process, \( G(s) \), is reduced.
  - Suppose the process (or plant) \( G(s) \) undergoes a change such that the true plant model is \( G(s) + \Delta G(s) \). The change in the plant may be due to a changing external environment or natural aging, or it may just represent the uncertainty in certain plan parameters.

Feedback and the effects of model error (or variation)

- We consider the effect on the tracking error \( E(s) \) due to \( \Delta G(s) \).
- Relying on the principle of superposition, we can let \( D(s) = N(s) = 0 \) and consider only the reference input \( R(s) \):
  \[
  E(s) + \Delta E(s) = \frac{1}{1 + (G(s) + \Delta G(s))C(s)}R(s)
  \]
  \[
  \Delta E(s) = \frac{1}{1 + (G(s) + \Delta G(s))C(s)}R(s) - E(s)
  \]
- Then the change in the tracking error is:
  \[
  \Delta E(s) = \frac{-C(s)\Delta G(s)}{(1 + G(s)C(s) + \Delta G(s)C(s))(1 + G(s)C(s))}R(s)
  \]
- Since we usually find that \( C(s)G(s) \gg C(s)\Delta G(s) \), we have:
  \[
  \Delta E(s) \approx -\frac{C(s)\Delta G(s)}{(1 + L(s))^2}R(s)
  \]
  We see that the change in the tracking error is reduced by the factor \( 1 + L(s) \), which is generally greater than 1 over the range of frequencies of interest.

Reduced sensitivity

- For large \( L(s) \), we have \( 1 + L(s) \approx L(s) \), and we can approximate the change in the tracking error by
  \[
  \Delta E(s) \approx -\frac{C(s)\Delta G(s)}{L(s)^2} = -\frac{1}{L(s)}\frac{\Delta G(s)}{G(s)}R(s)
  \]
- Larger magnitude \( L(s) \) translates into smaller changes in the tracking error (that is, reduced sensitivity to changes in \( \Delta G(s) \) in the process).
- Also, larger \( L(s) \) implies smaller sensitivity, \( S(s) \).
Reducing system sensitivity

• The question arises, how do we define sensitivity? Since our goal is to reduce system sensitivity, it makes sense to formally define the term. The system sensitivity is defined as the ratio of the percentage change in the system transfer function to the percentage change of the process transfer function.

• The system transfer function is:

\[ T(s) = \frac{Y(s)}{R(s)} \]

• and therefore the sensitivity is defined as:

\[ S = \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)} \]

• In the limit, for small incremental changes, \( S \) becomes:

\[ S = \frac{\delta T(s)/T(s)}{\delta G(s)/G(s)} = \frac{\delta \ln T(s)}{\delta \ln G(s)} \]

System Sensitivity.
System sensitivity is the ratio of the change in the system transfer function to the change of a process transfer function (or parameter) for a small incremental change.

• The sensitivity of the open-loop system to changes in the plant \( G(s) \) is equal to 1. The sensitivity of the closed-loop can be readily obtained.

• The system transfer function of the closed-loop system is:

\[ T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} \]

• Therefore, the sensitivity of the feedback system is:

\[ S_T = \frac{\delta T(s)/T(s)}{\delta G(s)/G(s)} = \frac{\delta \ln T(s)}{\delta \ln G(s)} \]

We find that the sensitivity of the system may be reduced below that of the open-loop system by increasing \( L(s) = G(s)C(s) \) over the frequency range of interest.

Parametric sensitivity

• Often, we seek to determine \( S_T^\alpha \), where \( \alpha \) is a parameter within the transfer function \( G(s) \).

• Using the chain rule, we find that:

\[ S_T^\alpha = S_T^G S_T^\alpha \]

• In such a case, we can write the transfer function of the system \( T(s) \) as a function of changing parameter \( \alpha \)

\[ T(s, \alpha) = \frac{\text{num}(s, \alpha)}{\text{den}(s, \alpha)} \]

• Then we may obtain the sensitivity to \( \alpha \) as:

\[ S_T^\alpha = \frac{\delta \ln T}{\delta \ln \alpha} = \left. \frac{\delta \ln \text{num}(s, \alpha)}{\delta \ln \alpha} \right|_{\alpha_0} - \left. \frac{\delta \ln \text{den}(s, \alpha)}{\delta \ln \alpha} \right|_{\alpha_0} = S_T^{\text{num}(s, \alpha)} - S_T^{\text{den}(s, \alpha)} \]

where \( \alpha_0 \) is the nominal value of the parameter \( \alpha \).

An important advantage of feedback control systems is the ability to reduce the effect of the variation of parameters of a control system by adding a feedback loop.

To obtain highly accurate open-loop systems, the components of the open-loop, \( G(s) \), must be selected carefully in order to meet the exact specifications. However, a closed-loop system allows \( G(s) \) to be less accurately specified, because the sensitivity to changes or errors in \( G(s) \) is reduced by the loop gain \( L(s) \).

This benefit of closed-loop systems is a profound advantage for the electronic amplifiers of the communication industry. A simple example will illustrate the value of feedback for reducing sensitivity.
An example of a feedback amplifier

- An amplifier used in many applications has a gain $-K_a$, as shown in Figure 5(a).
- The output voltage is
  \[ v_0 = -K_a \cdot v_{in} \]
- We often add feedback using a potentiometer $R_p$, as shown in Figure 5(b).
- The transfer function of the amplifier without feedback is:
  \[ T = -K \]
- and the sensitivity to changes in the amplifier gain is:
  \[ S_T K_a = 1 \]

- The closed-loop transfer function of the feedback amplifier is:
  \[ T(s) = \frac{-K_a}{1 + K_a \beta} \]
- The sensitivity of the closed-loop feedback amplifier is:
  \[ S^F_{K_a} = S^O_{K_a} = \frac{1}{1 + K_a \beta} \]
- If $K_a$ is large, the sensitivity is low. For example, if
  \[ K_a = 10^4 \]
- and
  \[ \beta = 0.1 \]
- we have:
  \[ S^F_{K_a} = \frac{1}{1 + 10^3} \]
  or the magnitude is one-thousandth of the magnitude of the open-loop amplifier.

5 Disturbance Signals in a Feedback Control System

Disturbance Signal Rejection

- An important effect of feedback in a control system is the control and partial elimination of the effect of disturbance signals.
- A disturbance signal is an unwanted input signal that affects the output signal. Many control systems are subject to extraneous disturbance signals that cause the system to provide an inaccurate output. Electronic amplifiers have inherent noise generated within the integrated circuits or transistors; radar antennas are subjected to wind gusts; and many systems generate unwanted distortion signals due to nonlinear elements. The benefit of feedback systems is that the effect of distortion, noise, and unwanted disturbances can be effectively reduced.
- We can compute the disturbance rejection by considering $R(s) = N(s) = 0$:
  \[ E(s) = - \frac{G(s)}{1 + L(s)}D(s) \]
  - For a fixed $G(s)$ and a given $D(s)$, as the loop gain $L(s)$ increases, the effect of $D(s)$ on the tracking error decreases. In other words, the sensitivity function $S(s)$ is small when the loop gain is large.
  - We say that large loop gain leads to good disturbance rejection. More precisely, for good disturbance rejection, we require a large loop gain over the frequencies of interest associated with the expected disturbance signals. In practice, the disturbance signals are often low frequency. When that is the case, we say that we want the loop gain to be large at low frequencies. This is equivalent to stating that we want to design the controller $C(s)$ so that the sensitivity function $S(s)$ is small at low frequencies.
An example I  The speed control for a steel rolling mill.

As a specific example of a system with an unwanted disturbance, let us consider the speed control system for a steel rolling mill. The rolls, which process steel, are subjected to large load changes or disturbances.

As a steel bar approaches the rolls (see Figure 6), the rolls are empty. However, when the bar engages in the rolls, the load on the rolls increases immediately to a large value. This loading effect can be approximated by a step change of disturbance torque.

An example II  The DC motor in open-loop.

Figure 6: The Steel Rolling Mill.

- Let \( R(s) = 0 \) and examine onlly the error \( E(s) = -\omega(s) \), for a given disturbance \( T_d(s) = D(s) \).
- The change in speed due to the load disturbance is then:

\[
E(s) = -\omega(s) = -\frac{1}{1 + \frac{b}{Js+K_aK_m}} T_d(s) = \frac{1}{b + Js + \frac{K_aK_m}{Js}} \cdot T_d(s)
\]

An example III  The steady state error.

- The steady-state error in speed due to the load torque, \( T_d(s) = \frac{D}{T} \), is found by using the final-value theorem.
- Therefore, for the open-loop system, we have:

\[
\lim_{t \to \infty} \epsilon(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{D}{(b + Js + \frac{K_aK_m}{Js})s} = \frac{D}{b + \frac{K_aK_m}{Js}}
\]

- The error, \( E(s) = -\omega(s) \), of the closed-loop system of Figure 8 is:

\[
E(s) = -\omega(s) = -\frac{G_2(s)}{(1+G_1(s)G_2(s)H(s))} T_d(s)
\]

where \( G_1(s) = \frac{K_aK_m}{Js} \), \( G_2(s) = \frac{1}{Js+K_a} \) and \( H(s) = K_t + K_b/K_a \).

- Then, if \( G_1(s)G_2(s)H(s) \) is much greater than 1, we obtain the approximate result:

\[
E(s) \approx \frac{1}{G_1(s)H(s)} T_d(s)
\]

- Therefore, if \( G_1(s)H(s) \) is made sufficiently large, the effect of the disturbance can be decreased by closed-loop feedback.
Sensitivity - Assignment 6.2

Consider the unity feedback system shown in the Figure 9. The system has two parameters, the controller gain $K$ and the constant $K_1$.

- Calculate the sensitivity of the closed-loop transfer function to changes in $K_1$.
- How would you select a value for $K$ to minimize the effects of external disturbances, $T_d(s)$?